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Some Properties of Transforms in Cultural Theory

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Abstract It is shown that, in certain circumstances, systems of cultural rules may be represented by doubly stochastic matrices denoted Π, called "possibility transforms," and by certain real valued "possibility densities" $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ with inner product $\langle \pi, \pi \rangle = \sum_i \pi_i^2 = 1$. We may characterize a certain problem of ethnographic or ethological description as a problem of prediction, in which observations are predicted by properties of fixed points of transforms of "pure systems", or by properties of convex combinations of such "pure systems". Other relationships to quantum methods are noted.

Keywords Quantum structures · Culture theory · Birkhoff theorem · Doubly stochastic matrices · Cultural rules · Ethnographic description · Ethnology · Ethology · MV-algebra · Possibility denisty

Background 1 This paper follows and adopts background from [4, 5] which we summarize here. We assume a finite non-empty set $\mathbf{P}$ whose members are called *individuals*, and a finite non-empty set $\mathbf{R}$ whose members are called *rules*. An *evolutionary structure* is a quintuple $\mathbf{S} := (\mathbf{P}, \mathbf{R}, D, B, M)$ where D, B, and M are binary relations on $\mathbf{P}$, satisfying these four axioms: (1) D is totally non-symmetric and transitive; (2) M is symmetric; (3) if bDc and there exists no $d \in \mathbf{P}$, $d \neq b, c$ for which bDd and dDc, then we write cPb, and require bBc iff for $b, c, d \in \mathbf{P}$, dPb and dPc; (4) $|bM| \leq 2$. A *rule* $R \in \mathbf{R}$, $\mathbf{R} \neq \varnothing$, is a statement concerning the relationships between the D, B, and/or M, which does not violate those four axioms. A family of subsets $\mathbf{G} = \{\mathbf{G}^t | \mathbf{G}^t \subseteq \mathbf{P}, t \in \mathbf{T}\}$ for $t \in \mathbf{T}$ a set of consecutive non-negative integers starting with 0, is called a *descent sequence* of $\mathbf{S}$. $\mathbf{G}^t$ is called a *generation* of $\mathbf{S}$, in case, for all $\mathbf{G}^t \in \mathbf{G}$ each cell bB occurs in only one generation, each subset bM occurs in only one generation, and for $t > 0$ when $\mathbf{G}^t \in \mathbf{G}$, $b \in \mathbf{G}^t$, and cPb, then $c \in \mathbf{G}^{t-1}$ (that is, the set $\mathbf{G}^t$ contains all of the immediate descendants of individuals in $\mathbf{G}^{t-1}$). We assume a "Darwinian Sequences axiom" which says that all descent sequences of a given evolutionary structure can be traced back through a chain of descent in an unbroken series of

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non-empty generations, to the same date of initial origin, which we assign as $t = 0$. We allow that only in such initial date may there be individuals in a generation that did not arise by descent from a previous generation of **S**. Only reproducing marriages are recognized—that is, all sets bM are assumed to be reproducing. (As we later discuss properties of smallest non-empty "configurations" among M and B relations, which "self-reproduce" under a rule, the assumption that marriages reproduce is not disturbing.) Therefore if **S** is an evolutionary structure and **G** is a descent sequence of **S**, then $\mathbf{B}^t := \{bB | b \in \mathbf{G}^t, \mathbf{G}^t \in \mathbf{G}, t > 0\}$ is a partition of $\mathbf{G}^t$; and $\mathbf{M}^{t-1} := \{bM | b \in \mathbf{G}^t, \mathbf{G}^t \in \mathbf{G}, t \geq 0\}$ is a set of subsets of $\mathbf{G}^{t-1}$; and $\mathbf{B}^t \to \mathbf{M}^{t-1}$ for $t \geq 1$ is a surjection.

Background 2 Given a generation $\mathbf{G}^t, a, b, c, \ldots, k \in \mathbf{G}^t$, $a \neq b \neq c \neq \cdots \neq k$, a *regular structure* is a closed cycle aBb, bMc, $cBd, \ldots, kMa$ of a finite number of alternating B and M relations, in which each $a \in \mathbf{G}^t$ occurs exactly twice in such a list, being exactly once on left of a B followed immediately by once on the right of an M, or once on the right of a B preceded immediately by once on the left of an M, or as the initial and the final symbol in such cycle, and in such cycle each $|bB| = |bM| = 2$. If there are j instances of M in such cycle, then the cycle is of type Mj. An ordered list counting the numbers of regular structures present in a particular $\mathbf{G}^t$ is a *configuration* $\xi_i := (m_0, m_1, \ldots, m_j, \ldots)$ where m_j is the number of regular structures of type Mj in ξ_i. Thus a configuration consisting only of 2 of the M2 structures would be written $(0, 0, 2, 0, \ldots)$. And $\mathbf{C} := \{\xi_i | \xi_i$ is a configuration, $i = 1, 2, \ldots, n\}$, $n > 0$ a finite positive integer, is a finite non-empty set of n distinct configurations. We also write $\xi \subseteq \mathbf{C}$. If ξ_i and ξ_j are configurations then $\xi_i + \xi_j$ is also a configuration, though $\xi_i + \xi_j \in \mathbf{C}$ is not required (since **C** is finite). We let $P(\mathbf{C}) := \{\xi \mid \xi \subseteq \mathbf{C}\}$ be a set of subsets of **C**.

Background 3 If ξ_i is a configuration on a generation $\mathbf{G}^t$ then $\mu(\xi_i) = |\mathbf{M}^t|$ or simply "μ", is the number of reproducing marriages in ξ_i; $\beta(\xi_i) = |\mathbf{B}^t|$ or simply β is the number of cells ("sibships") bB found in ξ_i; and γ is the total population of the generation $\mathbf{G}^t$ on which ξ_i is formed. If **G** is a descent sequence and, for $t \in T$, $\mathbf{G}^t \in \mathbf{G}$ is a generation in **G**, μ^t is the number of marriages on $\mathbf{G}^t$, and β^t the number of sibships on $\mathbf{G}^t$, then $\mu^{t-1} = \beta^t$ since $\mathbf{B}^t$ is a partition of $\mathbf{G}^t$, each $bM \in \mathbf{M}^{t-1}$ is assumed reproducing, each $bB \in \mathbf{B}^t$ must arise from just one $bM \in \mathbf{M}^{t-1}$, and $\mathbf{B}^t \to \mathbf{M}^{t-1}$ is a surjection. If μ is an integer then $\mathbf{C}_\mu := \{\xi_i | \xi_i \in \mathbf{C}$ and $\mu(\xi_i) = \mu\}$ is a *set of configurations of order* μ, and $P(\mathbf{C}_\mu)$ is a set of subsets of $\mathbf{C}_\mu$.

Background 4 Let **C** be a finite non-empty set of $n > 0$ configurations, n a positive integer, and let $P(\mathbf{C})$ be a set of subsets of **C**. Then let $\xi \in P(\mathbf{C})$ correspond to a "row vector" $\chi = (\chi_1, \chi_2, \ldots, \chi_n)$ where $\chi_i = 1$ if $\xi_i \in \xi$, and $\chi_i = 0$ otherwise, called the *content list* of ξ. We let $\Xi := \{\chi \mid \xi \in P(\mathbf{C}), \chi$ the contents list of $\xi\}$ be a set of contents list of $P(\mathbf{C})$. For example, if $n = 3$ so that $\mathbf{C} = \{\xi_1, \xi_2, \xi_3\}$, and if $\xi = \{\xi_1, \xi_2\}$ then the corresponding content list $\chi = (1, 1, 0)$. The transpose of a list $\chi = (\chi_1, \chi_2, \ldots, \chi_n)\}$ is denoted $\chi^T = (\chi_1, \chi_2, \ldots, \chi_n)^T$, so χ and χ^T represent the same list of configurations. Let **G** be a descent sequence and let $\mathbf{G}^t, \mathbf{G}^{t+1} \in \mathbf{G}$. For $\boldsymbol{\alpha} \in \mathbf{H}$, we write $\boldsymbol{\alpha} = [\alpha_{ij}]$, $0 \leq i, j \leq k$, where $\alpha_{ij} = 1$ if $\boldsymbol{\alpha}$ allows ξ_i on $\mathbf{G}^t$ to create ξ_j on $\mathbf{G}^{t+1}$ otherwise $\alpha_{ij} = 0$.

Definition 1 Let **C** be a finite non-empty set of $n > 0$ configurations, n a positive integer. Let **R** be a non-empty set of rules. For $\boldsymbol{\alpha} \in \mathbf{R}$, write $\boldsymbol{\alpha} = [\alpha_{ij}]$, $0 \leq i, j \leq k$, where $\alpha_{ij} = 1$ if $\boldsymbol{\alpha}$ allows ξ_i on $\mathbf{G}^t$ to create ξ_j on $\mathbf{G}^{t+1}$ otherwise $\alpha_{ij} = 0$. Let $\boldsymbol{\rho}, \boldsymbol{\sigma} \in \mathbf{R}$, let $\boldsymbol{\rho} = [\rho_{ij}]$ and $\boldsymbol{\sigma} = [\sigma_{ij}]$, let $\rho_i = (\rho_{i1}, \ldots, \rho_{in})$ be the ith row of $\boldsymbol{\rho}$ and $\sigma_j = (\sigma_{1j}, \ldots, \sigma_{nj})$ be the

jth column of $\boldsymbol{\sigma}$, where $0 \le i, j \le n$. Then $\alpha_{ij} := \rho_i \sigma_j = (\rho_{i1}\sigma_{1j} \circ \rho_{i2}\sigma_{2j} \circ \cdots \circ \rho_{iv}\sigma_{ij})$, where $xy = \min\{x, y\}$, and $x \circ y = \max\{x, y\}$. Then $\boldsymbol{\sigma\rho} = \boldsymbol{\alpha} = [\alpha_{ij}]$ is the *transform* of $\boldsymbol{\alpha}$. Applying this same "multiplication", if $\xi, \phi \in P(\mathbf{C})$ have contents lists χ and η respectively, we can write the product of $\boldsymbol{\alpha}$ and χ as $\boldsymbol{\alpha}\chi^T = \eta$ when $\boldsymbol{\alpha}$ applied to a set of configurations ξ produces the set ϕ.

Background 5 Definition 1 simply incorporates the results of [4], which also shows action of such transform on a set of configurations is an MV-algebra [4], and is associative. If $\mathbf{R}$ is a finite non-empty set of transforms on $\mathbf{C}$, then $\mathbf{H} = (\mathbf{R} \cup \{\boldsymbol{\alpha} \mid \boldsymbol{\alpha} = \boldsymbol{\rho\sigma} \text{ for } \boldsymbol{\rho}, \boldsymbol{\sigma} \in \mathbf{R}\})$ is a set of *transforms generated by* $\mathbf{R}$, so $\boldsymbol{\alpha} \in \mathbf{H}$ is a *history*, in particular is a *history generated by* $\mathbf{R}$, and $\mathbf{H}$ is a set of histories. Thus from Definition 1, a history is a rule, and a set of rules a set of histories, so an evolutionary structure can be described as $\mathbf{S} = (\mathbf{P}, \mathbf{H}, D, B, M)$, and henceforth we use only $\mathbf{H}$.

Background 6 Let $\mathbf{C}$ be a finite non-empty set of $n > 0$ configurations, n a positive integer. If $\boldsymbol{\alpha}$ is a history and there exists non-empty $\xi \subseteq \mathbf{C}$ with contents list χ such that $\boldsymbol{\alpha}\chi^T = \chi$; then $\boldsymbol{\alpha}$ is *viable* and we also say that $\boldsymbol{\alpha}$ is *viable on* ξ. A *minimal structure* of $\boldsymbol{\alpha}$ is a non-empty configuration $\xi_i \in \xi \subseteq \mathbf{C}$ such that $\boldsymbol{\alpha}$ is viable on ξ, and if there exists a nonempty configuration $\varphi_j \in \varphi \subseteq \mathbf{C}$, and $\boldsymbol{\alpha}$ is viable on φ then $\mu(\xi_j) \le \mu(\varphi_j)$. If ξ_j is a minimal structure of $\boldsymbol{\alpha}$, then $\mu(\xi_j) = s$ is the *structural number* of $\boldsymbol{\alpha}$. Denote a set of minimal structures of $\boldsymbol{\alpha}$ by $\mathbf{M}_{\boldsymbol{\alpha}}$. Then all minimal structures of a given history have the same structural number. A history $\boldsymbol{\alpha}$ acting only on a minimal structure of $\boldsymbol{\alpha}$ is viable. Let $\mathbf{C}$ be a non-empty set of configurations, and let $\mathbf{H}$ be a set of all transforms on $\mathbf{C}$ which are allowed by the rules of construction of configurations. Then we say that $\mathbf{H}$ is a *full* set of transforms on $\mathbf{C}$. If $\mathbf{H}$ is a full set of transforms on $\mathbf{C}$ and $\boldsymbol{\alpha}$ is viable on $\xi \subseteq \mathbf{C}$, then $\boldsymbol{\alpha} \in \mathbf{H}$.

Background 7 Let $\mathbf{C}$ be a finite non-empty set of configurations, let $\xi, \phi \in \mathbf{C}$ have contents list χ and η respectively; then each $\chi_i \in \chi$ and each $\eta_i \in \eta$ is a number either 0 or 1, so if $\chi, \eta \ne 0$, and if $\boldsymbol{\alpha}\chi^T = \eta$ then $\sum \chi_i \ge 1$ and $\sum \eta_i \ge 1$. If $\sum \eta_i = 1$ then $\boldsymbol{\alpha}$ has exactly one possible outcome, and if $\sum \chi_i = 1$ then $\boldsymbol{\alpha}$ acted on a set consisting of only one initial configuration. We wish to describe the "relative possibilities" when more than one possible outcome might occur, including outcomes when more than one immediately previous configuration might have existed, and we do not know which intermediate configuration actually occurred.

Definition 2 Let $\mathbf{C}$ be a finite non-empty set of $n > 0$ configurations, n a positive integer, let $\mathbf{H}$ be a finite non-empty set of histories on $\mathbf{C}$, and let $\boldsymbol{\alpha} \in \mathbf{H}$.

2.1. A *possibility transform of* $\boldsymbol{\alpha}$ is a matrix $\Pi = [p_{ij}]$ in which $0 \le p_{ij} \le 1$, $\sum_j p_{ij} \le 1$, $p_{ij} > 0$ iff $\alpha_{ij} = 1$. Then $\boldsymbol{\Pi} := \{\Pi \mid \Pi \text{ is a possibility transform}\}$ is a set of possibility transforms.
2.2. Let $\boldsymbol{\Pi}$ be a non-empty set of possibility transforms of dimension $n > 0$, let $\Pi, \Theta \in \boldsymbol{\Pi}$, and let $\chi, \eta \in \Xi$, where $\sum_i \chi_I = w \le n$ (respectively $\sum_i \eta_I = x \le n$). Let the products $\Pi\Theta$ be computed using ordinary arithmetic product of two matrices, and the products $\chi\Pi$ (respectively $\Pi\chi^T$) as the arithmetic product of a vector and a matrix. Noticing that $\sum_i \chi_I = w$, $w \le n$, then the *possibility density* π of $\chi\Pi$ (respectively $\Pi\chi^T$) is $\pi := (\pi_1, \pi_2, \ldots, \pi_n)$ (respectively $\pi^T := (\pi_1, \pi_2, \ldots, \pi_n)^T$) where $\pi_I = \sum_j (p_{ij}\chi_j / w)$ (respectively $\pi_I = \sum_j (p_{ji}\pi_j / w)$.
2.3. If π, ω are possibility densities then $\langle \pi, \omega \rangle = \sum_i (\pi_i \omega_i)$ is the *inner product* of π and ω.

Comment 1 Let $\mathbf{C}$ be a finite non-empty set of configurations, let $\xi \subseteq \mathbf{C}$ have content list χ. Since each $\chi_j = 0$ or $\chi_j = 1$, and each $p_{ij}\chi_j > 0$ iff $\chi_j = 1$, then if $\chi_j = 1$, $p_{ij}\chi_j = p_{ij}$, otherwise $p_{ij} = 0$. So while Definition 2.1 allows $\sum_j (p_{ij}\chi_j) \leq 1$, $\sum_j (p_{ij}\chi_j) = 1$ iff $\sum_j p_{ij} = 1$, which implies $\sum \chi_j > 0$. Thus a possibility density π of $\chi\Pi$ answers "if we know the transition ends in one of the non-empty configurations of ξ and the descent sequence applies the possibility transform Π, what if any are the non-empty configurations from which it may have started, and the relative possibility of each?" A possibility density ω^T of $\Pi\chi^T$ answers "if we know the transition starts in one of the non-empty configurations of ξ and the descent sequence applied the possibility transform Π, what are the non-empty configurations from which the transition might end and the relative possibility of each?" The inner product answers both questions at once. Given these interpretations, it is reasonable to impose this axiom:

Axiom 1 *If $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is a possibility density then $\sum_i \pi_i \leq 1$.*

Since a viable history must have a non-empty descent sequence in each generation, we would thus like to know when $\langle \pi, \omega \rangle = 1$.

Theorem 1 *Under conditions of Definitions 1 and 2 and notational conventions of the preceding Comments, if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{H}$, and $\Pi, \Theta \in \boldsymbol{\Pi}$ are possibility transforms of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ respectively, $\chi, \eta \in \Xi$, π the possibility density of $\chi\Pi$, ω^T the possibility density of $\Theta\phi^T$, $\sum_i \chi_i = w$, all these lists and matrices have dimension n, then $\langle \pi, \omega \rangle = 1$ iff all of:* (i) $\eta \neq 0$; (ii) $\chi \neq 0$; (*iii*) *for each i, $\sum_j p_{ij} = 1$ and $\sum_j q_{ij} = 1$*; (iv) $\eta = \chi$; *and also* (v) $\sum \chi_j = \sum \eta_j = w$.

Proof The products $\chi\Pi$ and $\Theta\eta^T$ are defined since the matrices and vectors all have the same dimension, and thus the respective possibility densities π, ω^T are defined. Note that $\langle \pi, \omega^T \rangle = \sum_i ((\sum_j (p_{ij}\chi_j/w)(\sum_j (q_{ij}\eta_j/w))$. Conditions (i) and (ii) are required since if either $\eta, \chi = 0$, $\langle \pi, \omega^T \rangle = 0$, since all of the products then involve at least one term $= 0$, thus all products $= 0$. Condition (iv) results since if there is any i such that $\eta_I = 1$ but $\chi_I = 0$, or any $\eta_I = 0$ but $\chi_I = 1$, then there will be a product $p_{ij}\chi_i q_{ji}\eta_i = 0$ in a numerator, but for which not both $p_{ij} = 0$ and $q_{ij} = 0$. But in that case even if $\sum_j p_{ij} = 1$ and $\sum_j q_{ij} = 1$, at least one of the $p_{ij} \neq 0$ or $q_{ij} \neq 0$ values will be disregarded in computing sums, so $\sum_i ((\sum_j (p_{ij}\chi_j/w)(\sum_j (q_{ij}\eta_j/w)) < 1$. Thus necessarily $\eta = \chi$ so (iv) is shown. To show conditions (iii) recall that $0 \leq p_{ij} \leq 1$ and $0 \leq q_{ij} \leq 1$, $\sum_j p_{ij} \leq 1$, $\sum_j q_{ij} \leq 1$, and thus every sum $\sum_j (p_{ij}\chi_i/w) \leq 1$, and every $\sum_j (q_{ij}\eta_i/w) \leq 1$. And if not both: (1) $\sum_j p_{ij} = 1$ for all i then there is at least one i such that $\sum_j (p_{ij}\chi_{ij}/w) < 1$, and (2) $\sum_j q_{ij} = 1$ for all i then there is at least one i such that $\sum_j (q_{ij}\eta_i/w) < 1$; so $\langle \pi, \omega^T \rangle = 1$ only if both $\sum_j p_{ij} = 1$ and $\sum_j q_{ij} = 1$. Thus (iv) is met, and since $\eta = \chi$ then $\sum \chi_j = \sum \eta_j = w$, so (v) is met. Thus only if (i) through (v) then $\langle \pi, \omega \rangle = 1$. But also by similar arguments, if $\langle \pi, \omega \rangle = 1$ then conditions (i) through (v) are met. □

Comment 2 If $\boldsymbol{\alpha}$ is viable then conditions (i), (ii), (iv) and (v) of Theorem 1 are met, but these conditions are requisite for (iii), so a history $\boldsymbol{\alpha}$ is viable if its transform Π meets Theorem 1. From Theorem 1(iv) $\eta = \chi$ and Comment 1, if any row or column of a possibility transform Π or Θ is that of some $\eta_i = \chi_i = 0$ then the corresponding $p_{ij} = 0$ and $q_{ij} = 0$. We can thus consider possibility transforms meeting Theorem 1 to be only their non-zero rows and columns, and the corresponding lists only the non-zero entries, all of dimension w; we call such forms *reduced*.

Lemma 1 *If* $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ *and* $\omega^T = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *are possibility densities meeting Theorem* 1, *then* $\sum_i \pi_i = 1$ *and* $\sum_i \omega_i = 1$.

Proof From Theorem 1(iii) each non zero row of Π is such that $\sum_j p_{ij} = 1$, where $0 \le p_{ij} \le 0$. From Definition 2.2, using the multiplication defined in Definition 2.3, and from Theorem 1(v), each ω is a convex combination of w rows of Π, each of which row is weighted by $1/w$. Thus, $\sum_i \omega_i = w(1/w)1 = 1$. By a similar argument, each $\sum_i \chi_i$ is a sum of exactly all non-zero entries p_{ij}, with exactly the same weights since there are w non-zero columns, and the denominators are all w, and, though possibly taken in a different sequence, that sum is thus given by $\sum_i \pi_i = \sum_i \omega_i = 1$. □

Definition 3 A *doubly stochastic matrix* is a matrix, each of whose rows and columns are non-negative numbers that sum to 1.

Theorem 2 *A possibility transform* Π *meeting Lemma* 1 *and Theorem* 1 *is doubly stochastic on those rows and columns for which* $\eta_i = \chi_i = 1$.

Proof Let Π be a possibility transform satisfying Theorem 1. We consider only rows and columns for which $\eta_i = \chi_i = 1$ (we need consider only the reduced form of Π). There are w such rows or columns. Theorem 1 establishes the result as to rows of Π. We extend the same reasoning as in Lemma 1. Given a non-empty set of configurations ξ, the possibility density ω is a sum of w columns, each weighted by $1/w$, whose sum is w, so the sum of each such column is $w(1/w) = 1$. □

Theorem 3 *Let* $\boldsymbol{\alpha}$ *be a history, let* $\chi \in \Xi$, *let* $\boldsymbol{\alpha}$ *be viable on* χ, *let be* Π *the possibility transform of* $\boldsymbol{\alpha}$, *let and* π *be the possibility density of* $\xi\Pi$ *then* $\langle \pi, \pi^T \rangle = \sum_i \pi_i^2 = 1$.

Proof Obvious. Set $\Theta = \Pi$, set $\phi = \chi$, then find the possibility densities π of $\chi\Pi$ and ω^T of $\Pi\xi^T$. But since Π is symmetric, $\pi^T = \omega^T$ so substituting in Theorem 1 gives $\langle \pi, \pi \rangle = 1$. Since $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, then also $\langle \pi, \pi \rangle = \sum_i \pi_i^2 = 1$. □

Comment 3 An important example of a set ξ of configurations meeting Theorems 1 through 3 is a set of the minimal structures of a transform (history, rule) $\boldsymbol{\alpha}$.

Definition 4 Let $\mathbf{C}_s$ be a non-empty set of configurations of order s. Let $\mathbf{H}_s$ be the full set of transforms on $\mathbf{C}_s$. Let $\boldsymbol{\Pi}_s := \{\Pi_\alpha \mid \boldsymbol{\alpha} \in \mathbf{H}_s, \Pi_\alpha$ is the possibility transform for $\boldsymbol{\alpha}\}$. Then $\mathbf{S}_s := (\mathbf{C}_s, \mathbf{H}_s, \boldsymbol{\Pi}_s)$ is a *cultural structure of order* s. And, if $\mathbf{S}_s$ is a cultural structure of order s, $\boldsymbol{\alpha}$ is a history with minimal structure $\xi_m \in \mathbf{C}_s$ and structural number s, $\mathbf{H}_s = \{\boldsymbol{\alpha}\}$, $\boldsymbol{\alpha} = [\alpha_{ij}]$, $\alpha_{ij} = 1$ iff $i = j = m$, then $\mathbf{S}_s$ is a *pure system* of $\boldsymbol{\alpha}$.

Comment 4 In a pure system of $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}$ is viable. Let $\boldsymbol{\alpha}$ be a history with structural number $s > 0$, and $\mathbf{S}_s$ a pure system of $\boldsymbol{\alpha}$, then $\mathbf{H}_s = \{\boldsymbol{\alpha}\}$. Since $\boldsymbol{\alpha}$ acts only on a particular minimal structure of $\boldsymbol{\alpha}$, then in a pure system $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{-1} = \boldsymbol{\alpha}^2$. In a pure system, the possibility transform Π of $\boldsymbol{\alpha}$ has but one entry $p_{ii} = 1$, where i indexes the minimal structure configuration ξ_m and all other $p_{jk} = 0$, $jk \ne ii$. Then for such Π, $\operatorname{tr}\Pi = 1$, indeed $\operatorname{tr}\Pi\Pi = 1$, and is obviously symmetric.

Definition 5 Under conditions of earlier definitions let $\mathbf{H}$ be a set of transforms, let $\boldsymbol{\alpha} \in \mathbf{H}$, let Π_α be the possibility transform of $\boldsymbol{\alpha}$, and let $\boldsymbol{\Pi} = \{\Pi_\alpha \mid \boldsymbol{\alpha} \in \mathbf{H}$, and Π_α is the possibility transform of $\boldsymbol{\alpha}\}$.

Let v_α be a real number $0 \le v_\alpha \le 1$ such that $\sum_\alpha v_\alpha = 1$. Then $\Psi := \sum_\alpha v_\alpha \Pi_\alpha$, is a *convex combination* of possibility transforms.

Comment 5 Let $\theta \subseteq \mathbf{H}$ be a non-empty subset of $\mathbf{H}$ such that $v_\alpha > 0$ iff $\boldsymbol{\alpha} \in \theta$. Now let $\Psi = [y_{ij}]$. Recall from Definition 2.1 that if $\boldsymbol{\alpha}$ is a history, the possibility transform of $\boldsymbol{\alpha}$ is $\Pi_\alpha = [p_{ij}]$ in which (i) $0 \le p_{ij} \le 1$, (ii) $\sum_j p_{ij} \le 1$, and (iii) $p_{ij} > 0$ iff $\alpha_{ij} = 1$. Clearly such Ψ meets (i) and (ii). The equalities occur in (ii) only if $\boldsymbol{\alpha}$ is viable. If we assume the set $\boldsymbol{\Pi}$ of Definition 4 includes only possibility transforms of pure systems, then clearly also, $\operatorname{tr}\Psi = 1$, and also $\operatorname{tr}(\sum_\alpha v_\alpha \Pi_\alpha) = \operatorname{tr}\Psi = 1$. Then for each such pure system, if $\Pi_\alpha = [p_{ij}] \in \theta$ there exists an $\boldsymbol{\alpha}$ such that exactly one $\alpha_{ij} = 1$. And therefore if $y_{ij} > 0$ there also exists at least one $\boldsymbol{\alpha}$ with at least one $\alpha_{ij} = 1$. This shows that when $\Psi = \sum_\alpha v_\alpha \Pi_\alpha$, $\boldsymbol{\alpha} \in \mathbf{H}$, $\Pi_\alpha \in \boldsymbol{\Pi}$, then also $\Psi \in \boldsymbol{\Pi}$. We may then need to construct a corresponding history $\boldsymbol{\alpha}$ (which is what anthropologists typically do in creating a minimally structured genealogy to describe the action of a "kinship terminology" in describing a "marriage system" [9]), but we know that at least one exists. At least many marriage and kinship systems are known to be described as permutations [10, 11]. Then since possibility densities of viable histories are doubly stochastic matrices, culture theory is an application of the "Birkhoff theorem" [7], that a set of doubly stochastic matrices of order n is the convex closure of the set of permutation matrices of the same order, and the vertices (extreme points) of that set are those permutations.

Interpretation Culture theory is often not seen as a science at all. But our results so far at least show that culture theory has some significant similarities to quantum methods. First, we can now structure ethnographic observation as a process of testing predictions. An ethnologist or ethologist observing a species of individuals in a descent sequence of some evolutionary structure at the outset hypothesizes that $\operatorname{tr}\Psi = 1$ meaning, that the descent sequence has some marriage rule(s), which can be characterized by some pure system(s). After observing, the ethnologist claims a subset of rules $\theta \subseteq \mathbf{H}_s$ such that $\operatorname{tr}\Psi = \operatorname{tr}(\sum_\alpha v_\alpha \Pi_\alpha) = 1$ for $\boldsymbol{\alpha} \in \theta$. Of course θ may consist of a single rule $\boldsymbol{\alpha}$ (for which therefore $v_\alpha = 1$). It is known [1–3], that if a descent sequence follows a rule $\boldsymbol{\alpha}$ with structural number s that certain measures on populations can be computed from the structural numbers of the rules thus observed. Thus the ethnologist can perform several simultaneous empirical observations on the descent sequence at date t, including: ask if the population of the generation observed claims at least one rule $\boldsymbol{\alpha}$ with structural number $s > 0$—that is, ask if the population of the generation(s) observed claim to use a non-empty set $\theta \subseteq \mathbf{H}$ such that $\operatorname{tr}\Psi = \operatorname{tr}(\sum_\alpha v_\alpha \Pi_\alpha) = 1$, for $\boldsymbol{\alpha} \in \theta$; and ask if the population measures observed are those predicted from the structural number or convex combination of structural numbers of the claimed rule(s). Ethnography thus asks which transform(s) $\boldsymbol{\alpha}$ have a fixed point that describes certain empirically observable and measurable population properties of a descent sequence following a claimed rule or set of rules. Comparison to "superposition" is thus present, similar to the idea of "mixed pure" systems as discussed by [6]: the combinations $\operatorname{tr}\Psi$ depict a single system simultaneously using rules with different structural numbers, resulting in physically testable predictions that are proven correct in [2, 3] and elsewhere. Social sciences such as demography (which studies eigenstates of age-structured birth and death matrices [13]) nor economics (which uses formalism of thermodynamics [12]) are unable to make such predictions. Our statistics of [1–3] do not use such thermodynamic forms, and even rest on a distinct combinatorial foundation [4]. A phase space interpretation also exists. Certain rules can be represented by minimal structures whose "size" (structural number) is the order of a group of permutations of labels on, and reflections of, those structures over time [10, 11]. As therefore a plane group, such permutat ions can be represented by "harmonic" forms on a single complex plane $\mathbb{C}$ and a single positive real dimension $\mathbb{R}^+$ ("time"). Thus, using minimal structures comprised from regular structures, cultural rules can be described on a phase space consisting of $\mathbb{C} \otimes \mathbb{R}^+$. This however is exactly the requirement for a massless quantum system [14] and, to our knowledge, cultural rules have no mass. And as noted, cultural rules require an MV-algebra [4].

Thus the status of culture theory is similar to that of physics of a century ago: much of what has not worked to provide a predictive theory is based on, or mathematically similar to, classical physics; but a restatement into a more "quantum" form which makes testable predictions, is possible. Thus expectation of broader application by authors such as [8, 14] may be justified.

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